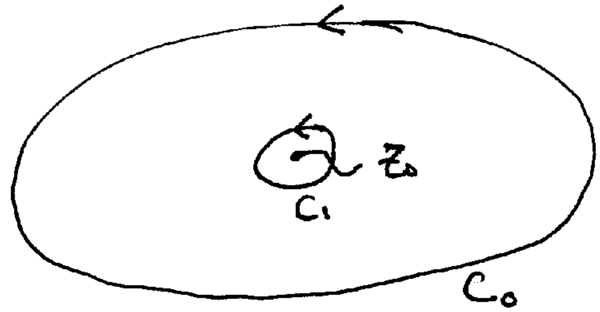
Multiply - Connected regions

Consider region where $f(z)$ is analytic in a domain bounded by C_0 and C_1

z_0 - may or may not be a singular pt.

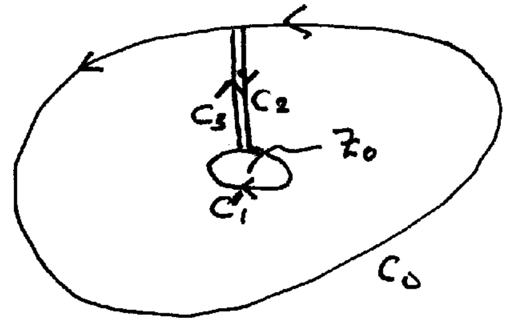


Cauchy-Goursat Theorem

$$\Rightarrow \quad \oint_{C_0} f(z)dz = \oint_{C_1} f(z)dz$$

Proof:

make a cut from C_0 to C_1



Now region enclosed by the new "curve" $C_1' C_3 C_0 C_2 \equiv C$ encloses a region in which $f(z)$ is analytic.

Apply Cauchy-Goursat Th to this region -

$$\oint_C f(z)dz = \int_{C_1'} f(z)dz + \int_{C_3} f(z)dz + \int_{C_0} f(z)dz + \int_{C_2} f(z)dz = 0$$

$\int_{C_1'} = -\oint_{C_1} f(z)dz$ - reverse direction of arrow on C_1'

Contributions on C_2 and C_3 are equal but opposite in sign $\rightarrow$ cancel

$$\therefore \oint_C f(z)dz = -\oint_{C_1} f(z)dz + \int_{C_0} f(z)dz = 0 \quad \Rightarrow \oint_{C_0} f(z)dz = \oint_{C_1} f(z)dz \quad \checkmark$$

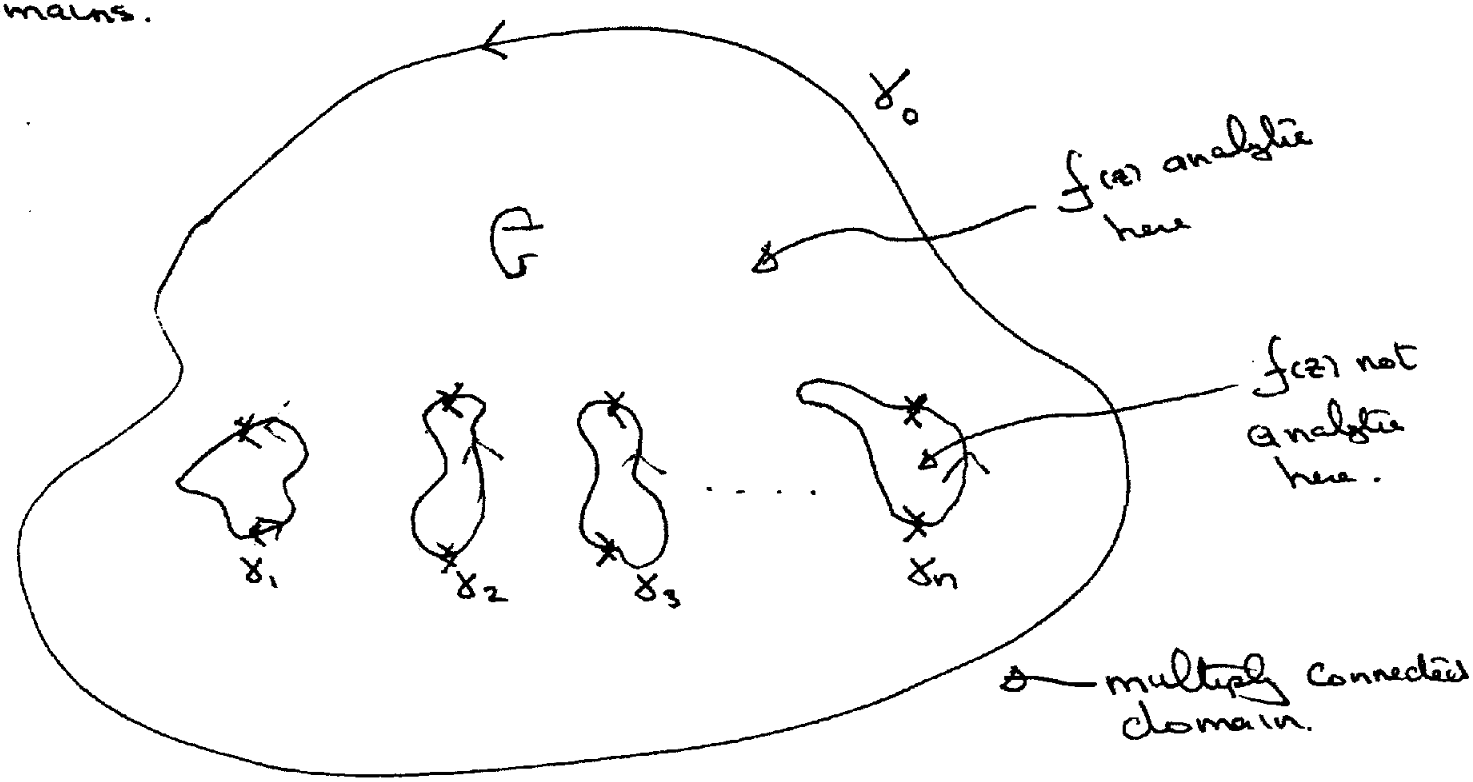
This result is important as one can continously shrink C_0 down onto C_1 without changing the result!

This is often referred to

This idea can be extended to multiple regions contained with C_0

- use γ_0 for C_0 on next pages.

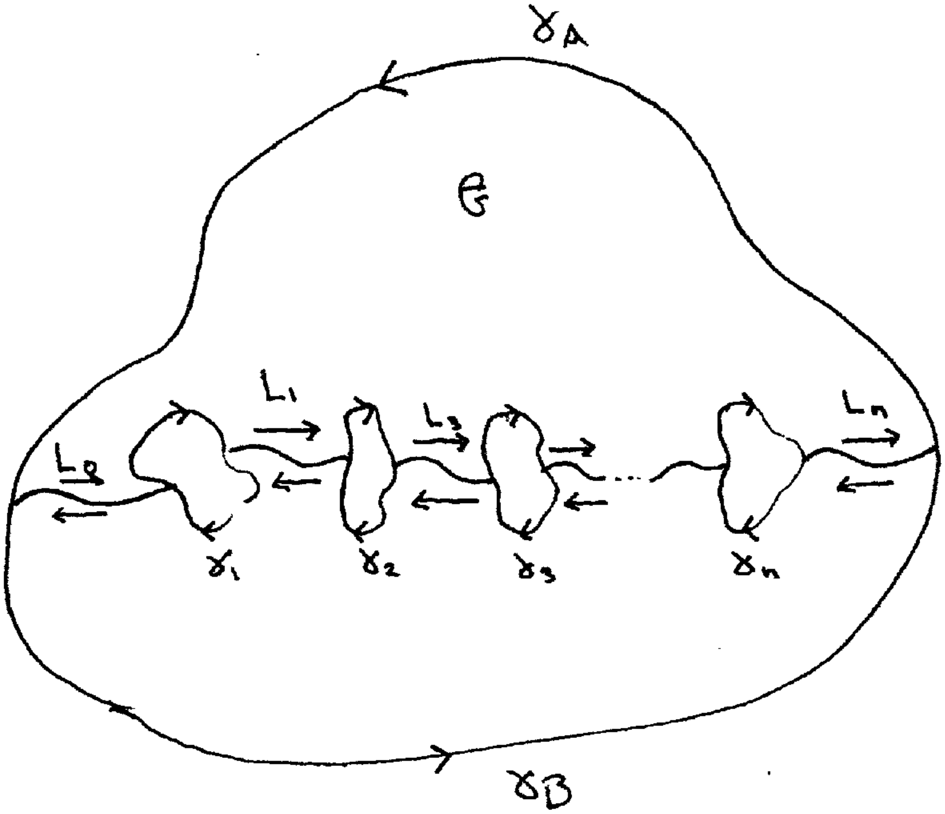
Extension of Cauchy's Theorem to multiply connected domains.



Theorem: Let the inside of the pws Jordan curve γ_0 contain the <u>disjoint</u> piecewise smooth Jordan curves $\gamma_1, \gamma_2, \ldots, \gamma_n$, none of which is contained inside another. Suppose $f(z)$ is analytic in a region G containing the set S consisting of all points on and inside γ_0 but <u>not</u> inside γ_k, $k=1,2,\ldots,n$. Then

$$\int_{\gamma_0} f(z)\,dz = \sum_{k=1}^{n} \int_{\gamma_k} f(z)\,dz .$$

Proof



Connect γ_k to γ_{k+1} with pws arcs L_k $k=0,1,\dots n-1$ and γ_n to γ_0 with L_n.

Note the reversal of path on γ_k, $k=1,2,\dots n$.

Now the upper and lower regions are decomposed into 2 simply connected domains where $f(z)$ is analytic $\therefore$ Cauchy's theorem is satisfied in each domain. (~~as~~ each curve is traversed in positive sense).

However, the integrals along L_k, $k=0,1,\dots n$ are carried out in oppose sense so these cancel out ($\int_{-C} f(z)dz = -\int_{C} f(z)dz$)

and the integrals over each γ_k, $k=1,2,\ldots n$ are the negative of the positively oriented integrals

$\therefore$ adding together the individual pieces

$$\int_{\gamma_0} f(z)\,dz - \sum_{k=1}^{n} \int_{\gamma_k} f(z)\,dz = 0$$

Note we have used in the proof

$$\int_{-C} f(z)\,dz = -\int_C f(z)\,dz$$

and
$$\int_{\gamma_1+\gamma_2} f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz$$

when γ_1, γ_2 are ~~piecewise~~ pws Jordan arcs.

This result is crucial $\longrightarrow$